Surgery on 3-Manifolds

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1. Preliminaries

Theorem 1.1. *Let $f : X \to Y$ be a function between topological spaces. Suppose that X and Y have the equivalence relations $\sim_X$ and $\sim_Y$, respectively, such that $x \sim_X x'$ if and only if $f(x) \sim_Y f(x')$. If f is a homeomorphism, then $X/\sim_X$ and $Y/\sim_Y$ are homeomorphic.*

Proof. Define a function $F : X/\sim_X \to Y/\sim_Y$ by $F[x] = [f(x)]$, where the square brackets denote equivalence classes. F is well defined since if $[x] = [x']$, then $x \sim_X x'$, thus $f(x) \sim_Y f(x')$ and $[f(x)] = [f(x')]$. We shall prove that F is a homeomorphism. To show that F is injective assume that $F[x] = F[x']$ so that $[f(x)] = [f(x')]$, i.e. $f(x) \sim_Y f(x')$. But then $x \sim_X x'$ and $[x] = [x']$. Surjectivity of F is easy to show. To prove that F is continuous, we consider the natural projections $\pi_X : X \to X/\sim_X$ and $\pi_Y : Y \to Y/\sim_Y$ which are continuous. Clearly, $F\pi_X = \pi_Y f$ and since f is continuous, we deduce that $\pi_X f$ is continuous and hence F is continuous by the universal mapping property of quotients. The fact that F^{-1} is continuous follows in a similar way because $F^{-1}\pi_Y = \pi_X f^{-1}$. □

Proposition 1.1 (Alexander trick)**.**

1. *Every orientation-preserving homeomorphism $f : S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\widetilde{f} : D^n \to D^n$.*
2. *Any two such extensions $\widetilde{f}_0, \widetilde{f}_1$ are isotopic through a family of homeomorphisms $\widetilde{f}_t$ such that $\widetilde{f}_t\big|_{S^{n-1}} = f$ for all $t \in [0,1]$.*

Proof.

(1) Let $f : S^{n-1} \to S^{n-1}$ be a homeomorphism. Consider $S^{n-1} = \partial D^n \subset \mathbf{R}^n$. Define the radial extension by

$$\widetilde{f} : D^n \to D^n, \qquad x \mapsto \widetilde{f}(x) := \begin{cases} \|x\| \, f\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is indeed a homeomorphism extending f.

(2) Let $\widetilde{f}_0$ and $\widetilde{f}_1$ be two homeomorphisms extending f. Define the homotopy $\widetilde{F} : D^n \times I \to D^n$ by

$$\widetilde{F}(x,t) := t\,\widetilde{f}_0(x) + (1-t)\,\widetilde{f}_1(x) \qquad \forall (x,t) \in D^n \times I.$$

Note that $\widetilde{f}_t = \widetilde{F}(\cdot, t)$ satisfies the desired condition.

□

The Jordan curve theorem states that if J is a simple closed curve in $\mathbf{R}^2$, then $\mathbf{R}^2 \setminus J$ has two components, and J is the boundary of each. Certainly, $\mathbf{R}^2$ can be replaced by S^2. Moreover, the Schönflies theorem ensures that the closure of one of the components of $\mathbf{R}^2 \setminus J$ is homeomorphic with the unit disk D^2. In the case of S^2, Schönflies theorem implies that the closures of both components of $S^2 \setminus J$ are 2-disks. This fact allows us to prove the following theorem.

Proposition 1.2. *Any two knots of S^1 in S^2 are equivalent.*

Proof. Let J_1 and J_2 be knots in S^2. For each $i \in \{1, 2\}$, let U_i and V_i be the two components of $S^2 \setminus J_i$. Since two knots are homeomorphic to S^1, there is a homeomorphism $h : J_1 \to J_2$. By the Alexander trick, h extends to a homeomorphism $h_U : \overline{U}_1 \to \overline{U}_2$ and also to $h_V : \overline{V}_1 \to \overline{V}_2$. Thus, by pasting lemma, we can define the homeomorphism $h : S^2 \to S^2$ by

$$h := \begin{cases} h_U & \text{in } \overline{U}_1, \\ h_V & \text{in } \overline{V}_1, \end{cases}$$

which carries J_1 to J_2. □

Proposition 1.3. *Any two links of $S^1 \cup S^1$ in S^2 are equivalent. In $\mathbf{R}^2$ there are two link types of $S^1 \cup S^1$ according as the first component is inside or outside the second.*

Proof. Let L_1 and L_2 be given links in S^2. Then $S^2 \setminus L_i$ has three components U_i, V_i and W_i, where $\overline{U}_i$, $\overline{W}_i$ are disks and $\overline{V}_i$ an annulus. Assume that the first component $L_i^{(1)}$ of L_i is $\overline{U}_i \cap \overline{V}_i$. Then a homeomorphism $h : L_1^{(1)} \to L_2^{(1)}$ extends to a homeomorphism $\overline{U}_1 \to \overline{U}_2$, then to a homeomorphism $\overline{U}_1 \cup \overline{V}_1 \to \overline{U}_2 \cup \overline{V}_2$, and finally to a homeomorphism $S^2 = \overline{U}_1 \cup \overline{V}_1 \cup \overline{W}_1 \to \overline{U}_2 \cup \overline{V}_2 \cup \overline{W}_2 = S^2$ taking L_1 to L_2 and respecting the ordering of the components. The $\mathbf{R}^2$ case is analogous. □

Proposition 1.4. *Any two knots in S^2 or $\mathbf{R}^2$ are ambient isotopic.*

Proof. Given two knots, there is a third knot disjoint from both of them, so it suffices to consider disjoint knots K_1, K_2 in (say) $\mathbf{R}^2$. There is a homeomorphism $h : \mathbf{R}^2 \to \mathbf{R}^2$ throwing $K_1 \cup K_2$ onto the circles of radius 1 and 2 centered at the origin. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be the piecewise-linear function defined by $f(x) =$

$(2x-1)\chi_{[1/2,1]}(x)+\frac{1}{2}(3-x)\chi_{(1,3]}(x)$. Define $g_t : \mathbf{R}^2 \to \mathbf{R}^2$ by $g_t(x) = (1+t\,f(|x|))\,x$. It is easy to see that this is an ambient isotopy and $g:1$ takes the circle of radius 1 to the circle of radius 2. Then $h^{-1}g_t h$ is an ambient isotopy which, when $t=1$, takes K_1 to K_2. The S^2 case follows from this since the isotopy is near ∞. $\square$

2. Obtaining 3-Manifolds by Surgery on S^3

An elementary r-surgery on a general n-manifold M is the operation of removing from M an embedded copy of $S^r \times D^{n-r}$ and replacing it with a copy of $D^{r+1} \times S^{n-r-1}$, the replacement being effected by means of a homeomorphism between the boundaries of the removed set and its replacement. Surgery in general is a sequence of elementary surgeries. In particular, we shall consider 1-surgeries on a 3-manifold, which certainly can be performed simultaneously, and consists of the removal from S^3 of disjoint copies of $S^1 \times D^2$ and their replacement by copies of $D^2 \times S^1$. Of course, the set removed and its replacement are homeomorphic, but the parameterization of the removed set as disjoint copies of $S^1 \times D^2$, and the canonical method of replacement with respect to that, ensure that the new manifold is usually not S^3. A collection of disjoint solid tori in S^3 is just a regular neighbourhood of a link, and a parameterization of a neighbourhood of each component by $S^1 \times D^2$ is called a *framing* of the link. Thus it will be shown that 3-manifolds can be interpreted by means of framed links in S^3.

Definition 2.1. *Piecewise linear homeomorphism h_0 and h_1 between complexes X and Y are isotopic if they are connected by a path of homeomorphisms $\{h_t : X \to Y\}_{t\in I}$ such that the map $H : X \times I \to Y \times I$ defined by $H(x,t) = (h_t(x), t)$ is a piecewise linear homeomorphism.*

Lemma 2.1. *Suppose that U and V are 3-manifolds with homeomorphic boundaries, and that $h_0 : \partial U \to \partial V$ and $h_1 : \partial U \to \partial V$ are isotopic homeomorphisms. Then $U \cup_{h_0} V$ and $U \cup_{h_1} V$ are homeomorphic.*

Proof. Choose a collar neighbourhood[1] C of ∂U in U. Let $\psi : C \to \partial U \times [0,1)$ be the homeomorphism associated with the collar. Let us define the function $f : U \sqcup V \to U \sqcup V$ by

$$f(x) = \begin{cases} x & \text{in} \quad (U \setminus C) \sqcup V, \\ \psi^{-1} F \psi(x) & \text{in} \quad C, \end{cases}$$

[1] If M is a manifold with boundary, then a collar neighbourhood of ∂M in M is an open neighbourhood homeomorphic to $\partial M \times [0,1)$ by a homeomorphism taking ∂M to $\partial M \times \{0\}$. See [Hat02, Proposition 3.42].

where $F : \partial U \times [0,1) \to \partial U \times [0,1)$ is given by $F(z,t) := (h_1^{-1}h_t(z), t)$. Since h_0 and h_1 are isotopic homeomorphisms, it follows that F it is as well. Moreover, $\psi^{-1}F\psi : C \to C$ is a homeomorphism. It is easy to see that, in this situation, f is a homeomorphism. Now, we prove that if $x \sim_{h_0} \tilde{x}$ (i.e. $h_0(x) = \widetilde{x}$), then $f(x) \sim_{h_1} f(\tilde{x})$ (i.e. $h_1(f(x)) = f(\widetilde{x})$). Let $x \in \partial U$ and $\widetilde{x} \in \partial V$ such that $h_0(x) = \widetilde{x}$. Then, $f(\widetilde{x}) = \widetilde{x}$, and

$$\begin{aligned} h_1(f(x)) = h_1(f(h_0^{-1}(\widetilde{x}))) =& h_1(\psi^{-1}F\psi(h_0^{-1}(\widetilde{x}))) = h_1(\psi^{-1}(h_1^{-1}h_0(h_0^{-1}(\widetilde{x})), 0)) \\ &= h_1(\psi^{-1}(h_1^{-1}(\widetilde{x}), 0)) = h_1(h_1^{-1}(\widetilde{x})) = \widetilde{x}, \end{aligned}$$

which proves that $h_1(f(x)) = f(\widetilde{x})$ as desired. Thus, f induces a homeomorphism in the quotient spaces. □

2.1. *Dehn twists*

In what follows, let F be a connected compact oriented surface, possibly with non-empty boundary. Let C be a simple closed curve embedded in F, and let A be an annulus neighbourhood of C. The standard annulus is $S^1 \times I$, with some fixed orientation.

Definition 2.2. *A twist about C is any homeomorphism isotopic to the homeomorphism $\tau : F \to F$ defined such that $\tau|_{F \setminus A}$ is the identity and, parameterizing A as $S^1 \times I$ in an orientation-preserving manner, $\tau|_A$ is given by $\tau(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$.*

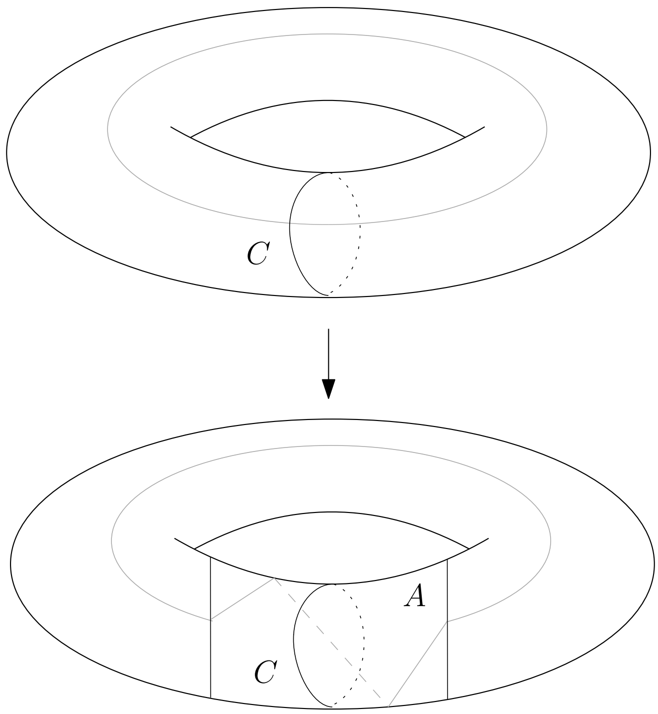


FIGURE 1. A twist about a curve C on a torus.

Definition 2.3. *Oriented simple closed curves p and q contained in the interior of the surface F are called twist-equivalent, written $p \sim_\tau q$, if $hp = q$ for some homeomorphism h of F that is in the group of homeomorphisms generated by all twists of F (which includes homeomorphisms isotopic to the identity).*

In this definition h is required to carry the orientation of one curve to that of the other. Of course, in general there may be no homeomorphism of any sort that sends p to q; that is certainly the case if p separates F and q does not.

Lemma 2.2. *Suppose oriented simple closed curves p and q, contained in the interior of the surface F, intersect transversely at precisely one point. Then $p \sim_\tau q$.*

Proof. See [Lic97, Lemma 12.5]. □

Lemma 2.3. *Suppose that oriented simple closed curves p and q contained in the interior of the surface F are disjoint and that neither separates F (that is, $[p] \neq 0 \neq [q]$ in $H_1(F, \partial F)$). Then $p \sim_\tau q$.*

Proof. By considering the surface obtained by cutting F along $p \cup q$, shows at once that there is a simple closed curve r in F that intersects each of p and q transversely at one point. Then, by the previous lemma, $p \sim_\tau r \sim_\tau q$. □

Proposition 2.1. *Suppose that oriented simple closed curves p and q are contained in the interior of the surface F and that neither separates F. Then $p \sim_\tau q$.*

Proof. See [Lic97, Proposition 12.7] □

Corollary 2.4. *Let $p_1, \dots, p_n$ be disjoint simple closed curves in the interior of F such that the union of which does not separate F. Let $q_1, \dots, q_n$ be another set of curves with the same properties. Then there is a homeomorphism h of F that is in the group generated by twists, so that $hp_i = q_i$ for each $i \in \{1, \dots, n\}$.*

Proof. Suppose inductively that such an h can be found so that $hp_i = q_i$ for each $i \in \{1, \dots, n-1\}$. Apply the previous proposition to hp_n and q_n in F cut along $q_1 \cup \dots \cup q_{n-1}$. □

2.2. *Handlebodies and Heegaard splitting*

Definition 2.4. *Let M be an n-manifold. For $0 \leq r \leq n$, an n-dimensional r-handle is a copy of $D^r \times D^{n-r}$, attached to the boundary of M along $\partial D^r \times D^{n-r}$ by an embedding $e : \partial D^r \times D^{n-r} \to \partial M$. The space $M \cup_e (D^r \times D^{n-r})$ is called "M with an r-handle added".*

Remark 2.1. *Note that the boundary of this new manifold is ∂M changed by an $(r-1)$-surgery.*

As in Figure 2, we shall call $D^r \times \{0\}$ the **core** of the handle, $\{0\} \times D^{n-r}$ the **cocore**, e the **attaching map**, $\partial D^r \times D^{n-r}$ the **attaching region**, $\partial D^r \times \{0\}$ the **attaching sphere** and $\{0\} \times \partial D^{n-k}$ the **belt sphere**. The number k is called the **index** of the handle.

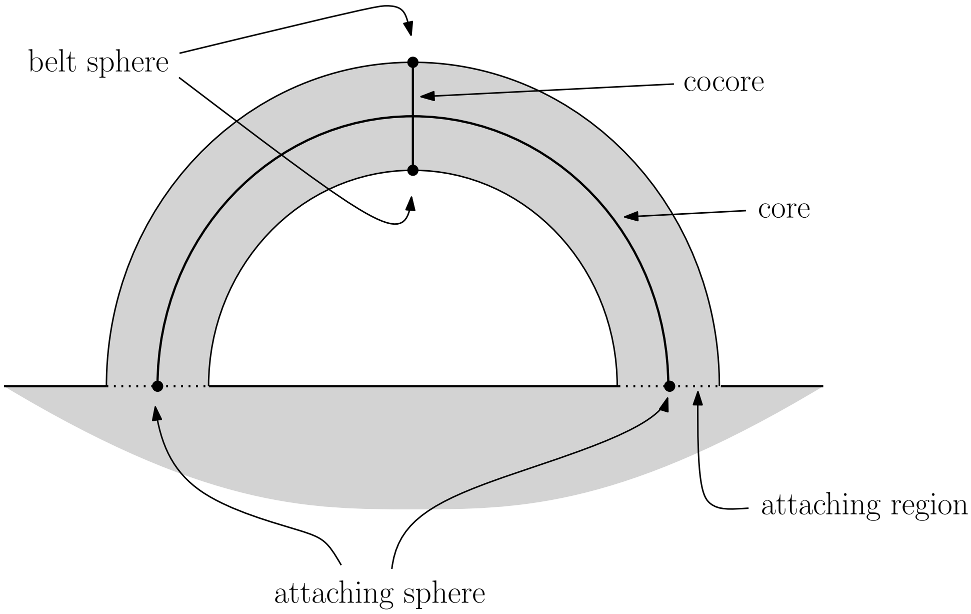


FIGURE 2. Anatomy of a handle.

Definition 2.5. *A handlebody of genus g is an orientable 3-manifold that is a 3-ball with g 1-handles added.*

Here, "orientable" can be taken to mean that every simple closed curve in the manifold has a solid torus neighbourhood. It is a straightforward exercise in the elementary technicalities of piecewise linear manifold theory to show that, up to homeomorphism, there is only one genus g handlebody. It is indeed, as already stated, the product of an interval with a g-holed disc. A regular neighbourhood of any finite connected graph embedded in an orientable 3-manifold is a handlebody. This follows by taking the neighbourhood of a maximal tree as the 3-ball and neighbourhood of the midpoints of the remaining edges as 1-handles.

Definition 2.6. *A Heegaard splitting of a closed connected orientable 3-manifold M is a pair of handlebodies X and Y contained in M such that $X \cup Y = M$ and $X \cap Y = \partial X = \partial Y$.*

Remark 2.2. *Note that X and Y have the same genus; namely, the genus of their common boundary surface.*

Lemma 2.5. *Any closed connected orientable 3-manifold has a Heegaard splitting.*

Proof. See [Lic97, Lemma 12.12] or [PS97, Theorem 8.3]. □

2.3. *Lickorish Theorem*

Theorem 2.6. *Let M be a closed connected orientable 3-manifold. There exist finite sets of disjoint solid tori $T'_1, \dots, T'_N$ in M and $T_1, \dots, T_N$ in S^3 such that $M \setminus \bigcup_{i=1}^N \operatorname{Int}(T'_i)$ and $S^3 \setminus \bigcup_{i=1}^N \operatorname{Int}(T_i)$ are homeomorphic.*

Proof. By Lemma 2.5, M has a Heegaard splitting, so for handlebodies U and V of some genus g, and some homeomorphism $h : \partial U \to \partial V$, $M = U \cup_h V$. Let $p'_1, \dots, p'_g$ be disjoint simple closed curves in ∂U, that bound disjoint discs in U and let $q_1, \dots, q_g$ be disjoint simple closed curves in ∂V (one around each "hole" of the handlebody) as shown in Figure 3.

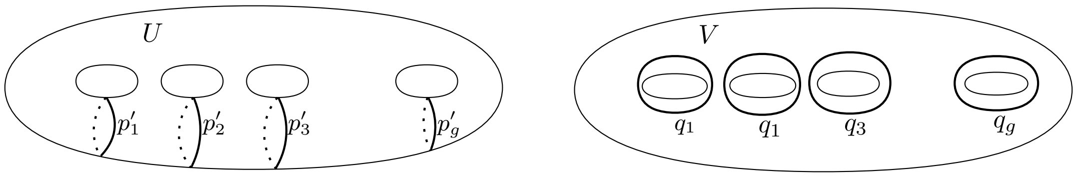


FIGURE 3

Let $h(p'_i) = p_i$ for each $i \in \{1, \dots, g\}$. If there were a homeomorphism $\psi : \partial V \to \partial V$ such that $\psi(p_i) = q_i$, then $\phi := \psi \circ h : \partial U \to \partial V$ would be a homeomorphism such that $\phi(p'_i) = q_i$, and, hence, $M \cong S^3$. This would complete the proof. If not, we can do an argument using twists. By Corollary 2.4, there is a product ψ, of twists and inverses of twists, of ∂V that sends each p_i to q_i. Up to isotopy a twist τ of ∂V is, by definition, supported on an annulus A. By Lemma 2.1 (and using the normality of the subgroup of all homeomorphisms isotopic to the identity) it may be assumed that all the twists concerned are so supported. Note that ∂V has a collar neighbourhood in V, a neighbourhood homeomorphic to $\partial V \times I$ with ∂V identified with $\partial V \times \{0\}$. Certainly, $A \times I \subset \partial V \times I$, and τ (initially supported on A) extends to $(\tau, \mathbb{1})$ on $A \times [0, 1/2]$. Then τ extends, by the identity, over the remainder of the closure of $V \setminus (A \times [1/2, 1])$. Observe that τ is not extended just in $\operatorname{Int}(A) \times (1/2, 1) \cong S^1 \times \operatorname{Int}(D^2) \cong \operatorname{Int}(S^1 \times D^2)$, which corresponds to the interior of a solid torus. We can conclude that τ extends over V after the removal of the interior of a solid torus. But this must be done for every twist in the product ψ, so, for this reason, we have to extend just in $A \times [0, 1/2]$ instead of $A \times I$, leaving space for the other twists. In other words, the solid tori that permit successive twists to extend are removed from successively narrower collars of ∂V. In summary, this means that the product ψ, of twists and inverse twists supported on annuli in ∂V, extends to a homeomorphism from V less the interiors of solid tori to V less the interiors of (in general, different) solid tori. Thus, at the cost of removing these solid tori, there is a homeomorphism of V to V sending each p_i to q_i, so gluing on copies of U by means of h to the first copy of V and by ψh to the second copy gives the required result. $\square$

Note that, with the notation of the above proof τ maps the boundary of the meridian disc of the solid torus $A \times [1/2, 1]$ to a curve that is homologous to a longitude, plus some number of meridians, on the boundary of the solid torus. Then, if the initial solid torus is parameterized as $S^1 \times D^2$, thus τ maps $\{\star\} \times \partial D^2$ (meridian) to $S^1 \times \{\star\}$ (longitude), which means that the boundary of the torus is affected by τ in exactly the same way as performing a 1-surgery on the manifold. This translates at once into the following result:

Theorem 2.7. *Any closed connected orientable 3-manifold M can be obtained from S^3 by a collection of 1-surgeries, that is, by removing disjoint copies of $S^1 \times D^2$ and replacing them with copies of $D^2 \times S^1$ in the canonical way. Thus M bounds a 4-manifold that is a 4-ball to which a collection of 2-handles has been added.*

In the application of this result, the disjoint copies of $S^1 \times D^2$ that are to be removed from S^3 are regarded as tubular neighbourhoods of a link embedded in S^3. To describe the identification of each neighbourhood with $S^1 \times D^2$, one must choose a parameterization in which the core $D^2 \times \{0\}$ of each solid torus corresponds to a component of the link. To specify such a parameterization, a simple closed curve, called a **framing curve**, is chosen on the boundary of each solid torus. This framing curve should bound a disc in the glued-in solid torus $D^2 \times S^1$ after the surgery is performed. Equivalently, the framing curve specifies how to glue $D^2 \times S^1$ back in.

Each framing curve can be determined by an integer assigned to the corresponding link component. This integer represents the **linking number** in S^3 between the component and its parallel copy (i.e. the framing curve), assuming both are oriented consistently and lie on the boundary of the solid torus neighbourhood.

An alternative way to describe a framed link is to represent each component as the core of a thin ribbon or band (topologically an annulus). The two boundary components of the annulus correspond to the link component and its parallel. When such a framed link is drawn in the plane (or more generally on a surface), the framing is often encoded visually by keeping the parallel close and consistently beside the link in the projection. This diagrammatic choice is commonly referred to as the *blackboard framing.*

The representation of a closed connected orientable 3-manifold by means of surgery on a framed link is by no means unique. That certainly seems likely from the proof of Theorem 2.6. There is no unique way of expressing a homeomorphism as a product of twists, for there are relations in the mapping class group of a surface. There is a theorem, due to Kirby, describing two ways in which the framed links can be changed without changing the 3-manifolds that result from them by means of surgery. It is fairly easy to see that the changes of links by such Kirby moves do not change the 3-manifold. What is not obvious is the fact that iterations of these two types of move relate any two framed links representing the same 3-manifold.

3. Surgery of 3-Manifolds

3.1. *Rational surgery along trivial knots*

Accordingly to the previous section, any compact orientable 3-manifold can be obtained from the sphere S^3 by several torus switches, i.e. 1-surgeries. In many situations it is more convenient to consider arbitrary reattaching homeomorphisms of the solid torus. The manifold obtained in this way is uniquely determined by the image J of the meridian α under the attaching homeomorphism of the boundary torus $S^1 \times S^1$. Suppose that $J = p\,\alpha + q\,\beta$, i.e. J is the closed curve that winds around the boundary torus p times along the meridian α and q times along the longitude β. More precisely, this means that the curve represents the element $(p, q) \in \mathbf{Z} \times \mathbf{Z} = \pi_1(S^1 \times S^1)$.

Proposition 3.1.

(1) *If the curve $p\,\alpha + q\,\beta$ is closed and has no self-intersections, then either the integers p and q are coprime, or one of them is 0 and the other is ± 1.*

(2) *If two closed curves without self-intersections on the torus are homotopic, then they are isotopic.*

Consider the surgery of the sphere S^3 in which a tubular ε-neighbourhood of the trivial knot $J \subset S^3$ is removed and the meridian α of the repasted solid torus is identified with the curve $p\alpha + q\beta$. Certainly, the surgery that glues α back onto $-\alpha$ does not change our manifold. Hence, surgery with the identification of the meridian and the curve $-p\alpha - q\beta$ is equivalent to surgery with the identification of the meridian and the curve $p\alpha + q\beta$.

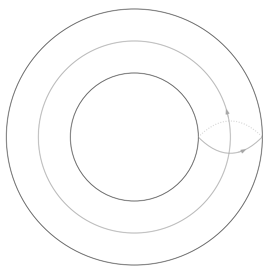


Figure 4. Meridian and longitude orientation.

We shall agree that the orientation of the meridian and longitude are chosen as shown in Figure 4.

By Proposition 3.1, surgery in S^3, as we are considering, is entirely determined by the rational number $r = p/q$. We shall call this number the **framing index** or simply **framing** of the trivial knot J, and the corresponding operation, rational surgery with framing index r.

Note that for identical surgery, we have $r = 1/0 := \infty$, while rational surgery of index $r = 0$ (along the trivial knot) is a torus switch (interchanges parallels and meridians).

In what follows $\bigcirc^r$ denotes the manifold obtained from the sphere S^3 by rational surgery along the framed trivial knot.

Proposition 3.2. $\bigcirc^0 = S^1 \times S^2$.

Proof. The sphere S^3 can be obtained by gluing together two solid tori T_1 and T_2 along a homeomorphism $f : \partial T_1 \to \partial T_2$ interchanging parallels and meridians. In turn, rational surgery with framing index $r = 0$ consists in cutting out the torus T_2 from S^3 and pasting it back in along a homeomorphism that exchanges parallels and meridians. The result is the gluing together of two solid tori T_1 and T_2 along the identical homeomorphism of their boundaries. Since $T_i \cong S^1 \times D^2$ and gluing together D^2 and D^2 along the identity map of their boundary circles produces S^2, gluing together T_1 and T_2 along the identity map of their boundaries produces $S^1 \times S^2$. □

Proposition 3.3. $\bigcirc^{p/q} = L(p, q)$.

Proof. The lens space $L(p, q)$ was obtained by gluing together two solid tori along the homeomorphism of their boundary that takes the meridian α of one torus to the curve $q\alpha + p\beta$ on the other torus. On the other hand, the sphere S^3 can be obtained by identifying the boundaries of these tori along the homeomorphism that takes α to β and β to α. Hence, $L(p, q)$ can be obtained from S^3 by regluing a solid torus along the homeomorphism that takes the curve α to $p\alpha + q\beta$. □

Proposition 3.4. $\bigcirc^{\pm 1/n} = S^3$.

Proof. If we use the definition of $L(p, q)$, then this proposition is a particular case of the previous one. Indeed, $L(p, q)$ is defined as the quotient of the unit 3-sphere in $\mathbf{C}^2$ by the equivalence

$$(z, w) \equiv \big(\exp(2\pi i/p)z\,,\, \exp(2\pi iq/p)w\big).$$

This definition implies that for $p = \pm 1$, no identifications of points occur, i.e. $L(\pm 1, n) = S^3$. □

Proposition 3.5. $\bigcirc^r \cong \bigcirc^{1/(\pm n + 1/r)}$.

Proof. It is clear from the definition of lens spaces that $L(p, q) \cong L(p, q \pm np)$, but this is exactly the statement of the proposition in a different notation. □

3.2. *Linking numbers*

Consider J and K two oriented curves in S^3. Let us pay attention only to those crossing points where the curve K passes over J. These crossing points can be of two types:

For each of the crossing points considered, take the corresponding value $\varepsilon_i = \pm 1$ and sum all these $\varepsilon = \pm 1$. The integer thus obtained is called the linking number of the curves J and K in S^3, and is denoted by $\operatorname{lk}(J, K)$.

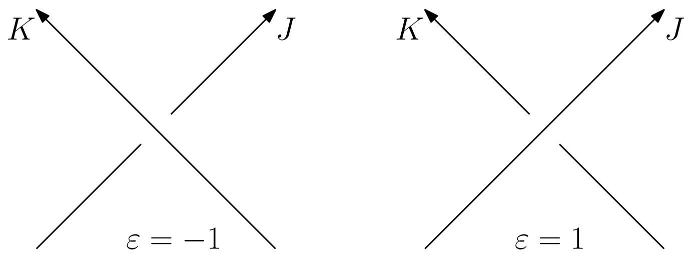


FIGURE 5. Two types of crossing points.

Theorem 3.1. *The linking number is an invariant of the link* $\{J, K\}$*, i.e. does not depend on the choice of the diagram of the link.*

Proof. It suffices to prove that $\operatorname{lk}(J, K)$ does not change under Reidemeister moves. The moves Ω_1 and Ω_3 never change the set of values of ε_i, while the move Ω_2 either does nothing, or adds (or destroys) two numbers ε_i of opposite signs. □

Two closed curves J and K in S^3 are called **unlinked** if there exists an isotopy that takes them to two curves J' and K' lying in two nonintersecting 3-disks. In the converse case, they are called **linked**. The linking number is a very simple invariant that allows one to prove (in many cases) that two curves are linked. Indeed, we have the following statement.

Theorem 3.2. *If* $\operatorname{lk}(J, K) \neq 0$*, then* J *and* K *are linked.*

Proof. Suppose the curves J and K are unlinked. Consider a diagram on which the projections of the 3-disks containing the curves J' and K' do not intersect. On this diagram the curves J' and K' have no common crossing points, and so $\operatorname{lk}(J', K') = 0$. Therefore, by the previous theorem, $\operatorname{lk}(J, K) = 0$. This contradicts the assumption of the theorem. Hence J and K are linked. □

3.3. *Meridian and longitude of arbitrary oriented knots*

For an arbitrary oriented knot J, there are no problems in choosing the meridian: we can simply take a section of the tubular ε-neighbourhood by a plane perpendicular to the knot at some point. However, the choice of the longitude presents some difficulties, because the natural form one can imagine depends on the diagram. In this way, we aim to give a definition of the longitude on the tubular ε-neighbourhood of J. First, we must exclude the situation on the left of Figure 6. To do so, we require that the curves J and K be codirected, i.e. if u and v are the orientation vectors on J and K at points of the same meridional disk, their scalar product (u, v) must be positive. A curve K on the boundary of the tubular ε-neighbourhood of J is said to be a longitude of J if K and J are codirected and $\operatorname{lk}(J, K) = 0$.

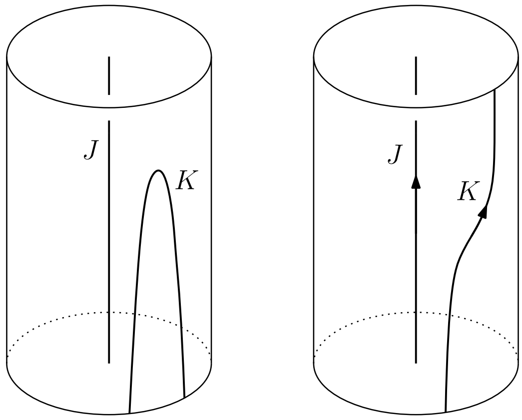


FIGURE 6. Not codirected and codirected curves.

3.4. *Integer surgery*

Now that we have agreed on the choice of the meridian and longitude, surgery on the sphere S^3 is uniquely determined by a knot J and its framing, a rational number r. Indeed, given (J, r), we begin by arbitrarily orienting the knot J, choose the meridian α and the longitude β, and orient them so that J and β are codirected and we have $\operatorname{lk}(\alpha, J) = +1$ (see Figure 7).

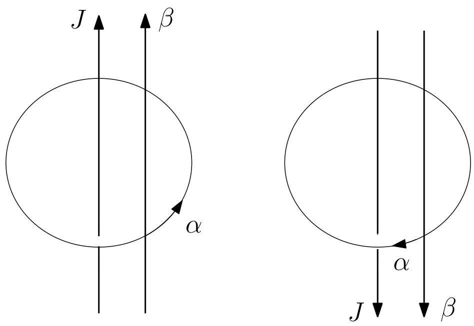


FIGURE 7. Orienting the longitude and meridian.

Under the homeomorphism that determines the surgery, the meridian α is taken to a curve of the form $p\alpha + q\beta$, where $p, q \in \mathbf{Z}$ are coprime. The integers p and q do not depend on the choice of the orientation of J, because if the orientation of J is reversed, then so is the orientation of α, so that we get the same curve. Thus to specify a surgery of the sphere S^3 it suffices to indicate a (nonoriented) knot diagram J and its framing $r = p/q$.

Theorem 3.3. *Any compact orientable 3-manifold without boundary can be obtained from the sphere S^3 by integer surgery.*

Proof. It is enough to note that one torus switch is an integer surgery along a trivial knot with framing 0, while a sequence of torus switches is an integer surgery along a link (consisting of unknotted components with framings equal to 0). □

3.5. *Surgery along ribbons*

Under integer surgery along a component J of the given link, the meridian α is mapped to the curve $K = p\alpha + \beta$. This curve effects exactly one revolution in the direction of the longitude β, so we can assume that the curve K and the knot J are codirected. In that situation they span a narrow ribbon (see Figure 8).

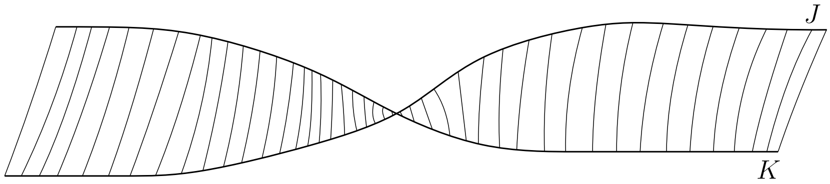


FIGURE 8. Ribbon spanning J and K.

Therefore, an integer surgery along J with framing $r = p$ is entirely determined by a ribbon with boundary components J and K, where

$$\mathrm{lk}\,(J, K) = \mathrm{lk}\,(J, p\alpha + \beta) = p,$$

since $\mathrm{lk}\,(J, \beta) = 0$ by definition of the longitude and $\mathrm{lk}\,(J, \alpha) = 1$. Moreover, it is not necessary to distinguish the boundary components, because $\mathrm{lk}\,(J, K) = \mathrm{lk}\,(K, J)$. Thus our integer surgery presentation is determined by a set of twisted ribbons (which may be knotted and linked). We call this method of defining integer surgery a *ribbon surgery presentation*.

3.6. *Equivalent surgeries*

Surgery on S^3 along different framed links can produce the same manifold (up to homeomorphism). Two such surgeries are called equivalent. We have already seen an example of equivalent surgeries performed along the circle (see Proposition 3.5). The sphere S^3 can be obtained by gluing the solid torus T_1—which is a tubular neighbourhood of the circle J—and the solid torus T_2 by a homeomorphism interchanging longitude and meridian (see Figure 9). Let us cut T_2 along the meridional disk D and perform a twist by n full turns. This changes the value of the framing r of J to $1/(\pm n + 1/r)$, where the sign in front of n depends on the direction of the twist.

In the case when the surgery is not only performed along the circle, but along a framed link containing the circle J as a component, the above homeomorphism establishes the equivalence of the surgeries represented in Figure 10. Here we assume that $r_2 \in \mathbf{Z}$ and $r_1 \in \mathbf{Q}$. The change in the framing r_2 is determined by the fact that the linking number of the boundary components of the ribbon, as can easily be seen, changes by ± 1 (recall that these components are codirected and reversal of their orientation does not affect their linking number). When we perform a twist by n turns, the framings r_1 and r_2 change to $r'_1 = 1/(\pm 1 + 1/r_1)$ and $r'_2 = \pm n + r_2$.

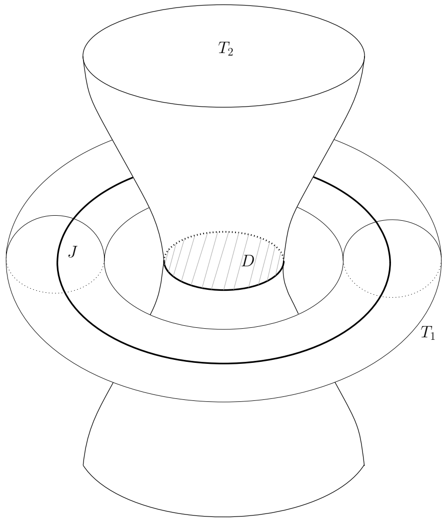
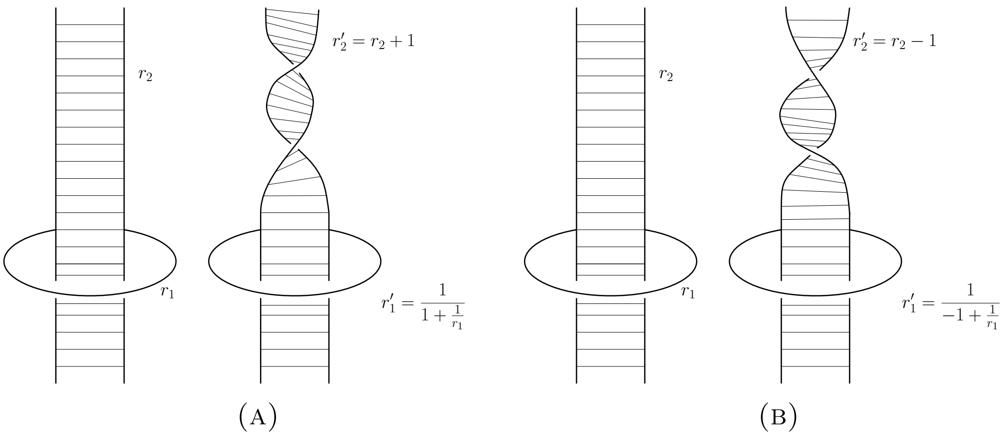


FIGURE 9



(A) (B)

FIGURE 10. Two equivalent surgeries.

3.7. *Homology spheres*

Definition 3.1. *A compact 3-manifold (without boundary) M^3 is said to be a* ***homology sphere*** *if its fundamental group $\pi_1 = \pi_1(M^3)$ coincides with its commutant*

$$\pi_1' = \left\{aba^{-1}b^{-1} : \quad a, b \in \pi_1\right\}.$$

Remark 3.1. *The equality $\pi_1' = \pi_1$ means that the quotient group π_1/π_1' consists only of the unit element.*

It can be proved that the homology groups of any homology sphere are the same as those of S^3. Let us also mention without proof that surgery of S^3 along any knot with framing ± 1 produces a homology sphere. Hence, it is not surprising that there are infinitely many nonhomeomorphic homology spheres, although surgery along different knots with framing ± 1 may produce the same homology sphere.

Among the various homology spheres, the most famous is the Poincaré homology sphere. Poincaré at first conjectured that any homology sphere is homeomorphic to S^3. But soon afterwards he himself constructed a counterexample, which we describe next, and used the fundamental group (which he had conveniently invented before that) to prove that it was not the true sphere. He then conjectured that any compact oriented 3-manifold (without boundary) with trivial fundamental group is homeomorphic to S^3 (the *Poincaré conjecture*).

The manifold obtained by surgery on the sphere S^3 along the right trefoil with framing 1 is called the *Poincaré homology sphere*, or briefly the *Poincaré sphere*. It is easy to verify that under symmetry in any plane the framing of a knot changes its sign. Therefore, the Poincaré sphere may also be obtained by surgering along the left trefoil with framing -1.

To prove that the Poincaré sphere is not S^3, we shall compute its fundamental group. We begin this computation by finding a presentation of $\pi_1(S^3 \setminus K)$, where K is the right trefoil. We assume that the base point O is at infinity. Any loop from O can clearly be represented as the composition of the loops x, y and z (see Figure 11), and their inverses.

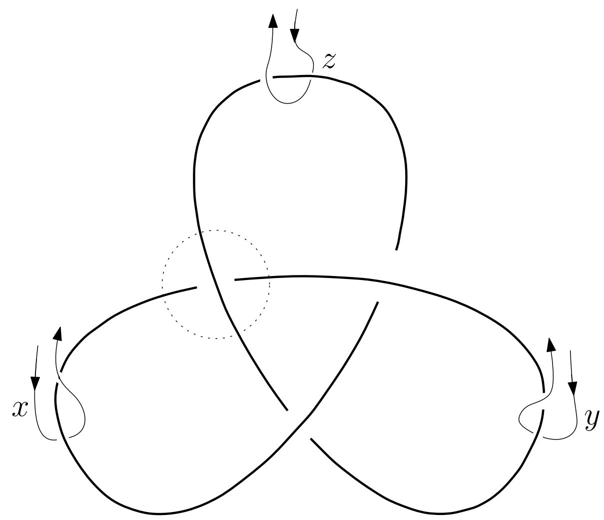


FIGURE 11. Generators of $\pi_1(S^3 \setminus K)$.

In other words, these loops generate the group $\pi_1(S^3 \setminus K)$. The crossing points yield the defining relations. For example, the crossing shown by the dotted circle in Figure 11 gives us $z = xyx^{-1}$, i.e. $zx = xy$.

Similarly, the two other crossings yield $xy = yz$ and $yz = zx$. Therefore,

$$\pi_1(S^3 \setminus K) = \langle x, y, z \mid xy = yz = zx \rangle.$$

Since $z = xyx^{-1}$, we can get rid of the generator z, obtaining the group

$$\pi_1(S^3 \setminus K) = \langle x, y \mid xyx = yxy \rangle.$$

When we perform surgery on S^3 along the right trefoil with framing 1, to the curve J (see Figure 12) we attach the spanning meridional disk of the solid torus.

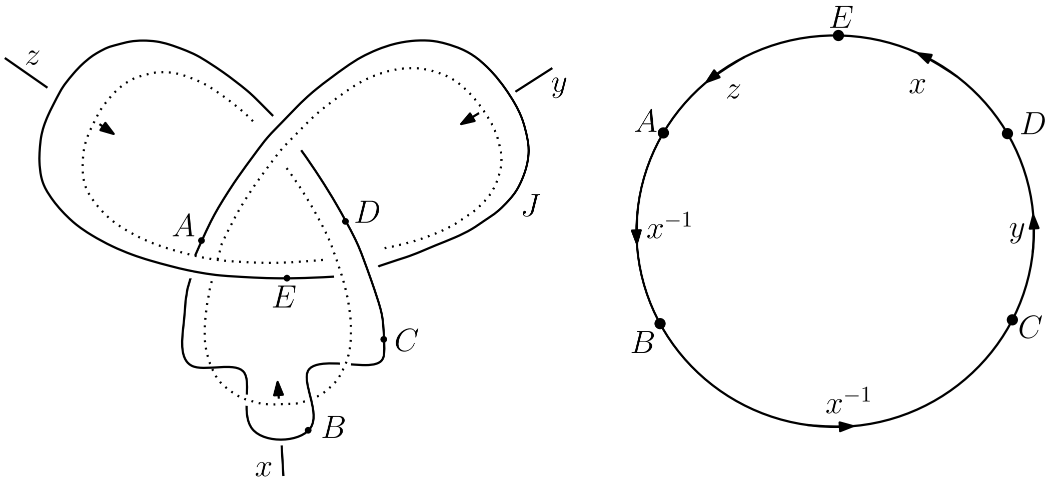


FIGURE 12. Relation in π_1 of the Poincaré sphere.

Hence, the fundamental group of the Poincaré sphere is obtained from the group $\pi_1(S^3 \setminus K)$ by adding the relation $J = 1$. It is easy to see (see Figure 12) that $J = x^{-2}yxz = x^{-2}yx^2yx^{-1}$. Thus the fundamental group of the Poincaré sphere is isomorphic to the group

$$I^\star = \langle x, y \mid xyx = yxy,\ yx^2y = x^3 \rangle.$$

It can be proved that $I^\star$ coincides with its commutant. Moreover, putting $a = x$ and $b = xy$, we have that

$$I^\star \cong \langle a, b \mid a^5 = b^3 = (ba)^2 \rangle,$$

which enables us to prove that the Poincaré sphere is not S^3. It is worth mentioning that $I^\star$ is known as the binary icosahedral group.

Certainly, the Poincaré sphere can be obtained by surgery along many links, not just the trefoil. Several such surgeries are presented in [PS97, Chapter VI, Section 18].

3.8. *The Kirby calculus*

Since the Poincaré sphere can be obtained through several different surgeries, it is natural now to study the transformations of framed links that do not alter the resulting manifold. These transformations constitute what is known as Kirby calculus.

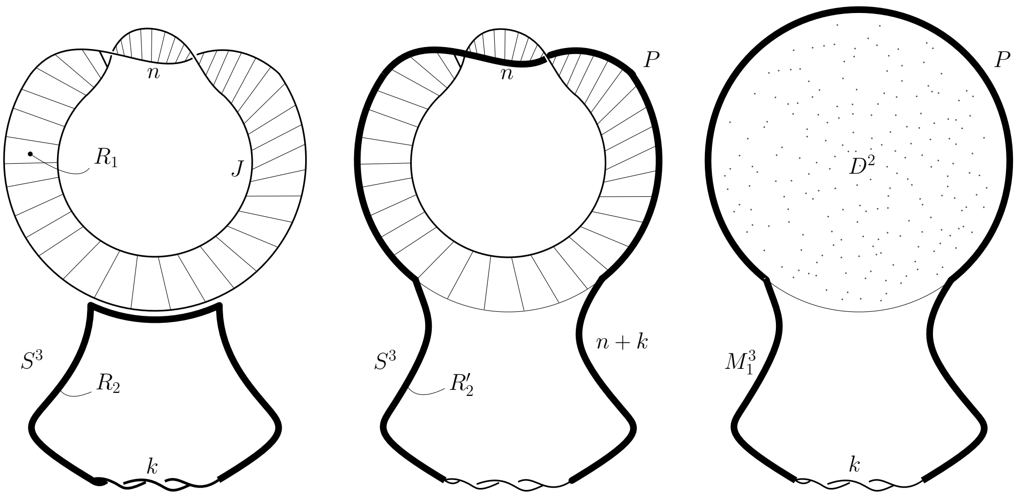
3.8.1. *The first Kirby move*

The Kirby move of the first kind consists in adding to (or deleting from) the given framed link $L \subset S^3$ an unknotted circle with framing ± 1 provided that it is unlinked with the other components of L, i.e. there exists a sphere S^3 enclosing the circle and bounding a 3-disk that does not intersect any other components of L. This transformation does not change the resulting 3-manifold, because when we perform surgery on S^3 along a circle with framing $\pm 1/n$, we get S^3 again.

In terms of ribbons, the first Kirby move consists in adding (deleting) a ribbon with one full twist (in either direction), provided the ribbon is not linked with the other ribbons in the ribbon presentation. For the ribbon presentation, as well as the integer surgery presentation, the first Kirby move may be described as adding a trivial component.

3.8.2. *The second Kirby move*

This transformation is easier to describe in the language of ribbons. We first consider the case when the ribbons R_1 and R_2 in S^3 are unlinked (Figure 13a). Suppose R_1 and R_2 have n and k twists, respectively. Let R_2' be the ribbon that coincides with R_2 except in the part P that encircles R_1, i.e. P rounds around R_1, remaining parallel to it and therefore performing n extra twists as compared with R_2 (Figure 13b). Then we claim that the surgery presented by the two ribbons $\{R_1, R_2\}$ produces the same manifold as that presented by the two ribbons $\{R_1, R_2'\}$ shown in Figure 13a and 13b.



(A) (B) (C)

FIGURE 13. Equivalent ribbon surgery presentations.

To prove that, first let us perform surgery along $R_1 \subset S^3$ (Figure 13b). Then, in M_1^3 the spanning disk D^2 is attached to the curve J in the boundary of the ribbon R_1 (Figure 13c). We can therefore slice the encircling part $P \subset R_2'$ through R_1 until it becomes the collar of D^2 and then slide it off D^2. In the process, the n extra twists of P disappear. Now the surgery presented by R_2' can be done in M_1^3 before or after the slide (which is an isotopy in M_1^3) with the same result. But the result of surgery of M_1^3 along R_2' after the isotopy is clearly the same as that of surgery of S^3 along R_1 and R_2.

In the construction described above, the assumption that the ribbons R_1 and R_2 are unlinked was only used to explicitly determine the framing of link component corresponding to R_2' from those corresponding to R_1 and R_2. The ribbon R_2' may be constructed (just as above) in the case when the ribbons R_1 and R_2 are arbitrarily linked. Of course, the topology (i.e. the number of twists) of the ribbon thus constructed defines the framing of the link component corresponding to R_2'.

The replacement of $\{R_1, R_2\}$ by $\{R_1, R_2'\}$ or viceversa is called a *Kirby move of the second kind*.

The second Kirby move can also be described in terms of framed links. In a given framed link with two distinguished unlinked components C and K with framing indices n and k (see 14a), respectively, the first of these components is modified, the curve C being replaced by the curve C' that differs from C only in that it encircles K (see Figure 14b) and in that its framing index is changed to $n + k$, the other components remaining the same. The general case can also be described, but, for the moment, we restrict ourselves to this case only.

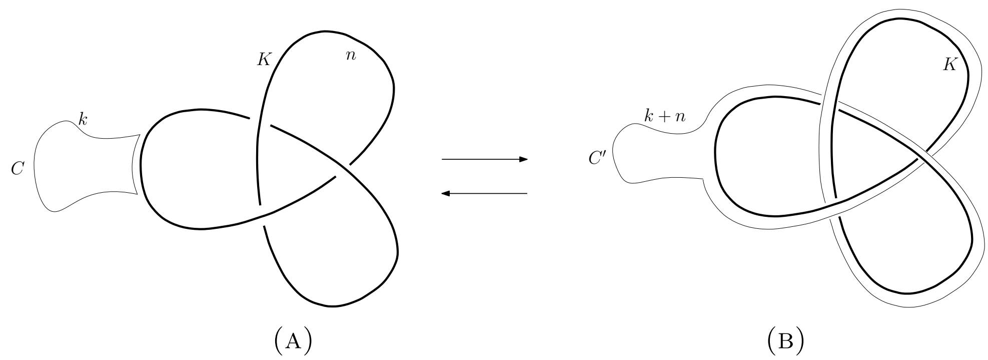


FIGURE 14. Example of the second Kirby move (for framed links).

We have shown that the second Kirby move, like the first one, does not change the result of the surgery. It turns out that the converse is also true.

Theorem 3.4 (Kirby's theorem). *Two links in S^3 with integer framings produce the same 3-manifold if and only if they can be obtained from each other by a finite sequence of Kirby moves of the first and second kinds and isotopies.*

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